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PARABOLIC PROBLEM IN A NON-CANONICAL DOMAIN DEGENERATING TO A POINT AT THE INITIAL MOMENT OF TIME

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Abstract. We study the boundary value problem for the two-dimensional heat equation in a non-canonical domain, the boundary of which changes according to the power law $x = t^{\omega}$, $\omega > \frac{1}{2}$. There is no domain of the solution of the problem at the initial time, that is, it degenerates into a point. Using the method of generalized heat potentials, the problem is reduced to a pseudo-Volterra integral equation of the second kind. The obtained integral equation is fundamentally different from the classical Volterra integral equations in that the norm of the corresponding integral operator is equal to one and the classical method of successive approximations is not applicable to it, and the corresponding homogeneous integral equation has a non-zero solution.

Keywords: heat equation, boundary value problem, degenerate domain, Volterra singular integral equation, regularization.

AMS Subject Classification: 45D05, 45E99.

1. Introduction

In recent years, with the accelerated development of modern technologies in the field of contact technique and increasing efficiency of electrical devices, there is a need for more accurate measurement of the temperature field of contact systems. It becomes especially important to study the dynamics of changes in this field over time. In this case, it is necessary to take into account the change in the size of the contact pad, which occurs under the influence of electrodynamic forces and at high temperatures due to the melting of the contact material.

To solve such boundary value problems, the Fourier method of separation of variables and the method of integral transformations are not applicable, since it is not possible to coordinate the solution of the heat equation with the motion of the boundary of the heat transfer domain. It is advisable to use generalized heat potentials and subsequent transformation of the original boundary value problem into pseudo-Volterra integral equations. The peculiarity of the problems under consideration is that, firstly, the domain in which solutions are sought has a moving boundary, and, secondly, at the initial moment of time, the contacts are in a closed state, which leads to the degeneration of the problem solution domain into a point [2-4, 6, 11-15, 17, 18, 23-27, 29].

In this paper, we study the case where the boundary of the domain, in which the solution of the problem is sought, moves according to the power law $x = t^{\omega}$ with $\omega > \frac{1}{2}$. Previous works have investigated heat conduction problems in conic domains that included both the inner and outer parts of a straight cone; that is, the domain degenerated to a point at the initial time instant of zero order, see the papers [1, 5, 7, 8, 10, 16, 19, 20, 22, 28]. The essential difference in this paper is that we examine the parabolic problem in regions where the solution domain degenerates into a point at the initial moment of time with a higher order of degeneration, since the generators of the cone are curves $x = t^{\omega}$ with $\omega > \frac{1}{2}$. In future work, we plan to continue

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studying this problem in the case where the boundary of the region changes according to an arbitrary function $x = \gamma(t)$ with $\gamma(0) = 0$.

2. Problem statement

We obtain the following boundary value problem: in the domain
$Q = \{(r,t)|\ 0 < r < t^{\omega}, 0 < t < T, \omega > 1/2\}$:

$$\frac{\partial u}{\partial t} = a^2 \cdot \frac{1-2\beta}{r} \cdot \frac{\partial u}{\partial r} + a^2 \cdot \frac{\partial^2 u}{\partial r^2}, \tag{1}$$

$$u(r,t)|_{r=0} = g_1(t), \quad t > 0, \tag{2}$$

$$u(r,t)|_{r=t^{\omega}} = g_2(t), \quad t > 0, \tag{3}$$

where $0 < \beta < 1$.

Such boundary-value problems in time-varying domains arise, for example, when mathematically modeling thermophysical processes in the electric arc of high-current disconnecting devices, where the effect of the axial section of the arc being constricted into a contact spot in the cathode region is considered. These problems are also relevant for developing new technologies in metallurgy, crystal production, laser technologies, and more.

3. Main result

Theorem 3.1. *Let the conditions $g_1(t) \in M(0,+\infty)$, $t^{\omega(1-\beta)}g_2(t) \in M(0,+\infty)$, $M(0,+\infty) = L_{\infty}(0,+\infty) \cap C(0,+\infty)$ are satisfied, then the boundary value problem (1)-(3) has a solution as:*

$$u(r,t) = \int_0^t \frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\bigg|_{\xi=\tau^{\omega}} \mu(\tau)d\tau + \int_0^t \frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\bigg|_{\xi=0} \nu(\tau)d\tau,$$

where

$$G(r,\xi,t-\tau) = \frac{1}{2a^2} \cdot \frac{r^{\beta} \cdot \xi^{1-\beta}}{t-\tau} \cdot \exp\left[-\frac{r^2+\xi^2}{4a^2(t-\tau)}\right] \cdot I_{\beta}\left(\frac{r\xi}{2a^2(t-\tau)}\right),$$

$$\nu(t) = 2a^2\beta g_1(t),$$

and $\mu(t)$ is determined from the following pseudo-Volterra integral equation:

$$\mu(t) - \int_0^t N(t,\tau)\mu(\tau)d\tau = f(t), \tag{4}$$

where

$$N(t,\tau) = \frac{t^{\omega\beta}\tau^{\omega(1-\beta)}(t^{\omega}-\tau^{\omega})}{2a^2(t-\tau)^2}\exp\left[-\frac{(t^{\omega}-\tau^{\omega})^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{t^{\omega}\tau^{\omega}}{2a^2(t-\tau)}\right] \cdot I_{\beta}\left(\frac{t^{\omega}\tau^{\omega}}{2a^2(t-\tau)}\right)$$
$$+\frac{t^{\omega(\beta+1)}\tau^{\omega(1-\beta)}}{2a^2(t-\tau)^2}\exp\left[-\frac{(t^{\omega}-\tau^{\omega})^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{t^{\omega}\tau^{\omega}}{2a^2(t-\tau)}\right] \cdot I_{\beta-1,\beta}\left(\frac{t^{\omega}\tau^{\omega}}{2a^2(t-\tau)}\right)$$
$$+\frac{t^{\omega\beta}(1-2\beta)}{(t-\tau)\tau^{\omega\beta}}\exp\left[-\frac{(t^{\omega}-\tau^{\omega})^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{t^{\omega}\tau^{\omega}}{2a^2(t-\tau)}\right] \cdot I_{\beta}\left(\frac{t^{\omega}\tau^{\omega}}{2a^2(t-\tau)}\right),$$

$$I_{\beta-1,\beta}(z) = I_{\beta-1}(z) - I_{\beta}(z),$$

$$f(t) = -2a^2 g_2(t) + 2a^2\widetilde{g_1}(t^{\omega},t),$$

$$\widetilde{g_1}(r,t) = \frac{1}{(2a^2)^{\beta}} \cdot \frac{1}{2^{\beta}} \cdot \frac{1}{\Gamma(\beta)} \int_0^t \frac{r^{2\beta}}{(t-\tau)^{\beta+1}} \cdot \exp\left[-\frac{r^2}{4a^2(t-\tau)}\right] \cdot g_1(t)d\tau,$$

where $I_{\nu}(z)$ – is a modified Bessel function of order ν.

4. Integral representation of the solution of the boundary value problem (1)-(3)

We seek the solution of the system given by (1)-(3) as a sum of the double layer heat potentials:

$$u(r,t)=\int\limits_0^t \left.\frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\right|_{\xi=\tau^\omega}\mu(\tau)d\tau+\int_0^t \left.\frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\right|_{\xi=0}\nu(\tau)d\tau,$$

where

$$G(r,\xi,t-\tau)=\frac{1}{2a^2}\cdot\frac{r^\beta\cdot\xi^{1-\beta}}{t-\tau}\cdot\exp\left[-\frac{r^2+\xi^2}{4a^2(t-\tau)}\right]\cdot I_\beta\left(\frac{r\xi}{2a^2(t-\tau)}\right),$$

where ξ – is a parameter, $0<\beta<1$, $I_\beta(z)$– is the modified Bessel function of order β (Infeld function) [21].

Calculating the derivative $\frac{\partial G(r,\xi,t-\tau)}{\partial \xi}$ and determining

$$\left.\frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\right|_{\xi=0}=\frac{1}{(2a^2)^{\beta+1}}\cdot\frac{r^{2\beta}}{2^\beta(t-\tau)^{\beta+1}}\cdot\frac{1}{\beta\Gamma(\beta)}\cdot\exp\left[-\frac{r^2}{4a^2(t-\tau)}\right],$$

and

$$\begin{aligned}\left.\frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\right|_{\xi=\tau^\omega}&=\frac{1}{4a^4}\frac{r^\beta\tau^{\omega(1-\beta)}}{(t-\tau)^2}\\&\times\exp\left[-\frac{r^2+\tau^{2\omega}}{4a^2(t-\tau)}\right]\left\{rI_{\beta-1}\left(\frac{r\tau^\omega}{2a^2(t-\tau)}\right)-\tau^\omega I_\beta\left(\frac{r\tau^\omega}{2a^2(t-\tau)}\right)\right\}\\&+\frac{1}{2a^2}\cdot\frac{r^\beta(1-2\beta)}{(t-\tau)\tau^{\omega\beta}}\exp\left[-\frac{r^2+\tau^{2\omega}}{4a^2(t-\tau)}\right]I_\beta\left(\frac{r\tau^\omega}{2a^2(t-\tau)}\right),\end{aligned}$$

we obtain the following representation for the solution of (1)-(3):

$$\begin{aligned}u(r,t)=\int\limits_0^t&\left\{\frac{r^\beta\tau^{\omega(1-\beta)}(r-\tau^\omega)}{4a^4(t-\tau)^2}\exp\left[-\frac{(r-\tau^\omega)^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{r\tau^\omega}{2a^2(t-\tau)}\right]I_\beta\left(\frac{r\tau^\omega}{2a^2(t-\tau)}\right)\right.\\&+\frac{r^{\beta+1}\tau^{\omega(1-\beta)}}{4a^4(t-\tau)^2}\exp\left[-\frac{(r-\tau^\omega)^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{r\tau^\omega}{2a^2(t-\tau)}\right]I_{\beta-1,\beta}\left(\frac{r\tau^\omega}{2a^2(t-\tau)}\right)\\&\left.+\frac{r^\beta(1-2\beta)}{2a^2(t-\tau)\tau^{\omega\beta}}\exp\left[-\frac{(r-\tau^\omega)^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{r\tau^\omega}{2a^2(t-\tau)}\right]\cdot I_\beta\left(\frac{r\tau^\omega}{2a^2(t-\tau)}\right)\right\}\mu(\tau)\,d\tau\\&+\int\limits_0^t\frac{1}{(2a^2)^{\beta+1}}\frac{r^{2\beta}}{2^\beta(t-\tau)^{\beta+1}}\frac{1}{\beta\Gamma(\beta)}\exp\left[-\frac{r^2}{4a^2(t-\tau)}\right]\cdot\nu(\tau)d\tau,\end{aligned}\tag{5}$$

where

$$t^{-\alpha+\omega(1-\beta)}\cdot e^{\frac{(2\omega-1)t^{2\omega-1}}{4a^2}}\cdot\mu(t)\in L_\infty(0,\infty),$$

for $\alpha=(2\omega-1)(1-\beta+\varepsilon)$, $0<\varepsilon<\beta$.

5. Reduction of the boundary value problem (1)-(3) to the pseudo-Volterra integral equation

Satisfying the boundary conditions (2)-(3) for the function $u(r,t)$ defined by the equality (5), we find one of the densities such that:

$$\nu(t)=2a^2\beta g_1(t)$$

and obtain the pseudo-Volterra integral equation for finding the second density given by:

$$\mu(t)-\int_0^t\Big\{\tfrac{t^{\omega\beta}\tau^{\omega(1-\beta)}(t^\omega-\tau^\omega)}{2a^2(t-\tau)^2}\exp\Big[-\tfrac{(t^\omega-\tau^\omega)^2}{4a^2(t-\tau)}\Big]\exp\Big[-\tfrac{t^\omega\tau^\omega}{2a^2(t-\tau)}\Big]I_\beta\Big(\tfrac{t^\omega\tau^\omega}{2a^2(t-\tau)}\Big)$$
$$+\tfrac{t^{\omega(\beta+1)}\tau^{\omega(1-\beta)}}{2a^2(t-\tau)^2}\exp\Big[-\tfrac{(t^\omega-\tau^\omega)^2}{4a^2(t-\tau)}\Big]\exp\Big[-\tfrac{t^\omega\tau^\omega}{2a^2(t-\tau)}\Big]I_{\beta-1,\beta}\Big(\tfrac{t^\omega\tau^\omega}{2a^2(t-\tau)}\Big) \tag{6}$$
$$+\tfrac{t^{\omega\beta}(1-2\beta)}{(t-\tau)\tau^{\omega\beta}}\exp\Big[-\tfrac{(t^\omega-\tau^\omega)^2}{4a^2(t-\tau)}\Big]\exp\Big[-\tfrac{t^\omega\tau^\omega}{2a^2(t-\tau)}\Big]I_\beta\Big(\tfrac{t^\omega\tau^\omega}{2a^2(t-\tau)}\Big)\Big\}\mu(\tau)d\tau=f(t),$$

where

$$f(t)=-2a^2g_2(t)+2a^2\widetilde{g_1}(t^\omega,t).$$

6. On the peculiarities of the original pseudo-Volterra integral equation (6)

To study the original integral equation (6) to which our boundary value problem has been reduced, we introduce a new function such as:

$$\mu_1(t)=\ t^{\omega(1-\beta)}\cdot\mu(t),\quad f_1(t)=\ t^{\omega(1-\beta)}\cdot f(t). \tag{7}$$

Then, we get

$$\mu_1(t)-\int_0^t\sum_{i=1}^3 N_{i\omega}(t,\tau)\cdot\exp\left[-\frac{(t^\omega-\tau^\omega)^2}{4a^2(t-\tau)}\right]\mu_1(\tau)d\tau=f_1(t), \tag{8}$$

where

$$N_{1\omega}(t,\tau)=\frac{(1-2\beta)t^\omega}{(t-\tau)\tau^\omega}\exp\left[-\frac{t^\omega\tau^\omega}{2a^2(t-\tau)}\right]\cdot I_\beta\left(\frac{t^\omega\tau^\omega}{2a^2(t-\tau)}\right),$$

$$N_{2\omega}(t,\tau)=\frac{t^{2\omega}}{2a^2(t-\tau)^2}\exp\left[-\frac{t^\omega\tau^\omega}{2a^2(t-\tau)}\right]I_{\beta-1,\beta}\left(\frac{t^\omega\tau^\omega}{2a^2(t-\tau)}\right),$$

$$N_{3\omega}(t,\tau)=\frac{t^\omega\left(t^\omega-\tau^\omega\right)}{2a^2(t-\tau)^2}\exp\left[-\frac{t^\omega\tau^\omega}{2a^2(t-\tau)}\right]I_\beta\left(\frac{t^\omega\tau^\omega}{2a^2(t-\tau)}\right).$$

Remark 6.1. *For any value $\omega>\frac{1}{2},\ 0<\beta<1$, the following equations are true:*

$$\lim_{t\to0}\int_0^t\{N_{1\omega}(t,\tau)+N_{2\omega}(t,\tau)\}d\tau\ =\frac{1-\beta}{\beta},\quad \lim_{t\to0}\int_0^t N_{3\omega}(t,\tau)d\tau\ =0,$$

$$\int_0^t N_{3\omega}(t,\tau)d\tau\le C(a,\omega)\cdot t^{\frac{2\omega-1}{2}}.$$

Remark 6.2. *It follows from Remark 6.1 that when $\frac{1}{2}<\beta<1$, the integral equation (5) in the class of essentially bounded functions has a single solution which can be found by the method of successive approximations. When $0<\beta<\frac{1}{2}$, this equation is a pseudo-Volterra equation and the method of successive approximations does not apply to it.*

Therefore, we next consider the case $0<\beta<\frac{1}{2}$ and in order to find the solution of the pseudo-Volterra integral equation (6), we construct the corresponding characteristic equation.

7. Characteristic integral equation

In order to solve the pseudo-Volterra integral equation (8), we construct the corresponding characteristic integral equation. When constructing this equation, it is essential to consider two key points. First, the kernels of the basic and characteristic integral equations should be closely related in that their difference exhibits only a weak singularity. Second, the solution of the characteristic equation must be obtainable explicitly. In our previous work [27], we examined the model case where $\omega = 1$, and there, the solution of the corresponding basic integral equation was found in explicit form. The characteristic integral equation corresponding to the Eqn.(8) is derived from the model integral equation through appropriate substitutions:

$$t_1 = \left(\frac{1}{2\omega-1}\cdot t\right)^{\frac{1}{2\omega-1}}, \quad \tau_1 = \left(\frac{1}{2\omega-1}\cdot \tau\right)^{\frac{1}{2\omega-1}}.$$

For the pseudo-Volterra equation (8), the characteristic integral equation is as follows:

$$\mu_1(t) - \int\limits_0^t \sum_{i=1}^{2} N_{ih}(t,\tau)\cdot e^{-\frac{(2\omega-1)\cdot\left(t^{2\omega-1}-\tau^{2\omega-1}\right)}{4a^2}}\mu_1(\tau)d\tau = q(t), \tag{9}$$

where

$$N_{1h}(t,\tau) = \frac{(1-2\beta)(2\omega-1)t^{2\omega-1}}{\left(t^{2\omega-1}-\tau^{2\omega-1}\right)\tau} e^{-\frac{(2\omega-1)\cdot t^{2\omega-1}\tau^{2\omega-1}}{2a^2\left(t^{2\omega-1}-\tau^{2\omega-1}\right)}} I_\beta\left(\frac{(2\omega-1)\cdot t^{2\omega-1}\tau^{2\omega-1}}{2a^2\left(t^{2\omega-1}-\tau^{2\omega-1}\right)}\right),$$

$$N_{2h}(t,\tau) = \frac{(2\omega-1)^2 t^{4\omega-2}\tau^{2\omega-2}}{2a^2\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^2} e^{-\frac{(2\omega-1)\cdot t^{2\omega-1}\tau^{2\omega-1}}{2a^2\left(t^{2\omega-1}-\tau^{2\omega-1}\right)}} I_{\beta-1,\beta}\left(\frac{(2\omega-1)\cdot t^{2\omega-1}\tau^{2\omega-1}}{2a^2\left(t^{2\omega-1}-\tau^{2\omega-1}\right)}\right).$$

Let us show that the Eqn.(9) is indeed characteristic for the Eqn.(6). First, we need to show that the following remark is true.

Remark 7.1. *For any value $\omega > \frac{1}{2}$, $0 < \beta < \frac{1}{2}$, the following equation is true:*

$$\lim_{t\to 0}\int\limits_0^t \{N_{1h}(t,\tau) + N_{2h}(t,\tau)\}d\tau = \frac{1-\beta}{\beta}.$$

Proof. Indeed,

$$\begin{aligned}
&\int\limits_0^t \{P_{1h}(t,\tau) + P_{2h}(t,\tau)\}d\tau \\
&= \int\limits_0^t \left\{ \frac{(1-2\beta)(2\omega-1)t^{2\omega-1}}{(t^{2\omega-1}-\tau^{2\omega-1})\tau} e^{-\frac{(2\omega-1)\cdot t^{2\omega-1}\tau^{2\omega-1}}{2a^2(t^{2\omega-1}-\tau^{2\omega-1})}} I_\beta\left(\frac{(2\omega-1)t^{2\omega-1}\tau^{2\omega-1}}{2a^2(t^{2\omega-1}-\tau^{2\omega-1})}\right)\right. \\
&\left. + \frac{(2\omega-1)^2 t^{4\omega-2}\tau^{2\omega-2}}{2a^2(t^{2\omega-1}-\tau^{2\omega-1})^2} e^{-\frac{(2\omega-1)\cdot t^{2\omega-1}\tau^{2\omega-1}}{2a^2(t^{2\omega-1}-\tau^{2\omega-1})}} I_{\beta-1,\beta}\left(\frac{(2\omega-1)t^{2\omega-1}\tau^{2\omega-1}}{2a^2(t^{2\omega-1}-\tau^{2\omega-1})}\right)\right\} d\tau \\
&= \left\| z = \frac{(2\omega-1)t^{2\omega-1}\tau^{2\omega-1}}{2a^2(t^{2\omega-1}-\tau^{2\omega-1})}, \quad dz = \frac{(2\omega-1)^2 t^{4\omega-2}\tau^{2\omega-2}}{2a^2(t^{2\omega-1}-\tau^{2\omega-1})^2} d\tau \right\| \\
&= \int\limits_0^\infty \left\{ \frac{(1-2\beta)(2\omega-1)t^{2\omega-1}}{(t^{2\omega-1}-\tau^{2\omega-1})\tau} \cdot \frac{2a^2(t^{2\omega-1}-\tau^{2\omega-1})^2}{(2\omega-1)^2 t^{4\omega-2}\tau^{2\omega-2}} \cdot \frac{(2\omega-1)t^{2\omega-1}\tau^{2\omega-1}}{2a^2(t^{2\omega-1}-\tau^{2\omega-1})} \cdot \frac{1}{z} e^{-z} I_\beta(z)\right. \\
&\left. + \frac{(2\omega-1)^2 t^{4\omega-2}\tau^{2\omega-2}}{2a^2(t^{2\omega-1}-\tau^{2\omega-1})^2} \cdot \frac{2a^2(t^{2\omega-1}-\tau^{2\omega-1})^2}{(2\omega-1)^2 t^{4\omega-2}\tau^{2\omega-2}} \cdot e^{-z} I_{\beta-1,\beta}(z)\right\} dz \\
&= \int\limits_0^\infty \left\{(1-2\beta)\cdot \frac{1}{z} e^{-z} I_\beta(z) + e^{-z}\left\{I_{\beta-1}(z) - I_\beta(z)\right\}\right\} dz = \frac{1-2\beta}{\beta} + 1 = \frac{1-\beta}{\beta}.
\end{aligned}$$

□

Secondly, it is necessary to show that the following theorem is true.

Theorem 7.1. *If $\omega > \frac{1}{2}$, $0 < \beta < \frac{1}{2}$, then the following inequality is true:*

$$|(P_{1\omega} + P_{2\omega}) - (P_{1h} + P_{2h})| \leq C_1(\omega, \beta)\,|\omega - 1| \cdot \frac{t^{(\beta+1)\omega-1} \cdot \tau^{(\beta-1)\omega}}{(t-\tau)^{\beta}} \cdot e^{-\frac{(t^{\omega}-\tau^{\omega})^2}{4a^2(t-\tau)}}.$$

Proof. We introduce the following notation:

$$P_{i\omega}(t,\tau) = N_{i\omega}(t,\tau) \cdot e^{-Q_{\omega}(t,\tau)}, \quad P_{ih}(t,\tau) = N_{ih}(t,\tau) \cdot e^{-Q_h(t,\tau)}, \quad i = 1, 2.$$

where

$$Q_{\omega}(t,\tau) = \frac{(t^{\omega} - \tau^{\omega})^2}{4a^2(t-\tau)}, \quad Q_h(t,\tau) = \frac{(2\omega-1)\cdot(t^{2\omega-1} - \tau^{2\omega-1})}{4a^2}.$$

□

To prove this theorem, we will show that the following lemmas are true.

Lemma 7.1. *If $\omega \geq 1$, $0 < \beta < \frac{1}{2}$, then the following inequality*

$$Q_h(t,\tau) \geq Q_{\omega}(t,\tau), \quad \text{that is} \quad e^{-Q_h(t,\tau)} \leq e^{-Q_{\omega}(t,\tau)},$$

holds. If $\frac{1}{2} < \omega < 1$, $0 < \beta < \frac{1}{2}$, then the following inequality

$$Q_h(t,\tau) \leq Q_{\omega}(t,\tau), \quad \text{that is} \quad e^{-Q_h(t,\tau)} \geq e^{-Q_{\omega}(t,\tau)}.$$

holds.

Proof. Let $\omega \geq 1$. Then, from [9, p. 39]

$$(2\omega-1)(t^{2\omega-1} - \tau^{2\omega-1})(t-\tau) = \left[(2\omega-1)t^{2\omega-2}(t-\tau)\right] \cdot \frac{t^{2\omega-1} - \tau^{2\omega-1}}{t^{2\omega-2}}$$
$$\geq (t^{2\omega-1} - \tau^{2\omega-1}) \cdot \frac{t^{2\omega-1} - \tau^{2\omega-1}}{t^{2\omega-2}} = \left[\frac{t^{2\omega-1} - \tau^{2\omega-1}}{t^{\omega-1}}\right]^2 = \left[\frac{t^{2\omega-1}}{t^{\omega-1}} - \frac{\tau^{2\omega-1}}{t^{\omega-1}}\right]^2$$
$$\geq \left[\tfrac{t^{2\omega-1}}{t^{\omega-1}} - \tfrac{\tau^{2\omega-1}}{\tau^{\omega-1}}\right]^2 = (t^{\omega} - \tau^{\omega})^2.$$

Let $\frac{1}{2} < \omega < 1$.

$$(2\omega-1)(t^{2\omega-1} - \tau^{2\omega-1})(t-\tau) = (2\omega-1)(t^{2\omega-1} - \tau^{\omega-1}\cdot\tau^{\omega})(t-\tau)$$
$$= \left\| \begin{array}{c} \tau < t \\ \tau^{1-\omega} < t^{1-\omega}, \ \ \frac{1}{2} < \omega < 1 \\ \tau^{\omega-1} > t^{\omega-1}, \ \ \frac{1}{2} < \omega < 1 \end{array} \right\| < (2\omega-1)(t^{2\omega-1} - t^{\omega-1}\cdot\tau^{\omega})(t-\tau)$$
$$= (2\omega-1)t^{\omega-1}\cdot(t^{\omega} - \tau^{\omega})(t-\tau) = \left\| \ \omega < 12\omega - 1 < \omega \ \right\|$$
$$< \omega t^{\omega-1}(t-\tau)(t^{\omega} - \tau^{\omega})(t^{\omega} - \tau^{\omega})\cdot(t^{\omega} - \tau^{\omega}) = (t^{\omega} - \tau^{\omega})^2.$$

□

The lemma is proved.

Lemma 7.2. *If $\omega > \frac{1}{2}$, $0 < \beta < \frac{1}{2}$, then the following inequality is true:*

$$|P_{1\omega} - P_{1h}| \leq C_i(\omega, \beta)\,|\omega - 1| \cdot \frac{t^{(\beta+1)\omega-1} \cdot \tau^{(\beta-1)\omega}}{(t-\tau)^{\beta}} \cdot e^{-Q_{\omega}(t,\tau)}.$$

Proof. Let $\omega > 1$. Then, we have

$$|P_{1\omega} - P_{1h}| \leq |N_{1\omega} - N_{1h}|\, e^{-Q_{\omega}} + N_{1h}e^{-Q_{\omega}} \left|1 - e^{-(Q_h - Q_{\omega})}\right|,$$

Let us first evaluate the expression – $|N_{1\omega} - N_{1h}|$,

$|N_{1\omega} - N_{1h}| = N_{1\omega}\left|1 - \frac{N_{1h}}{N_{1\omega}}\right| = \left\|e^{-z}\cdot I_{\beta}(z) \approx \frac{1}{\sqrt{2\pi z}}; \quad z \gg 1\right\|$

$$\leq N_{1\omega}(t,\tau)\cdot\left|1-\frac{(2\omega-1)t^{2\omega-1}}{\tau}\cdot\frac{\tau^{\omega}}{t^{\omega}}\cdot\frac{t-\tau}{t^{2\omega-1}-\tau^{2\omega-1}}\cdot\left[\frac{t^{\omega}\tau^{\omega}}{t-\tau}\cdot\frac{(t^{2\omega-1}-\tau^{2\omega-1})}{(2\omega-1)t^{2\omega-1}\cdot\tau^{2\omega-1}}\right]^{\frac{1}{2}}\right|$$

$$=N_{1\omega}(t,\tau)\cdot\left|1-\frac{(2\omega-1)^{\frac{1}{2}}t^{2\omega-1-\omega+\frac{\omega}{2}-\omega+\frac{1}{2}}}{\tau^{1-\omega-\frac{\omega}{2}+\omega-\frac{1}{2}}}\cdot\lim_{\substack{\tau\to t\\ \tau=tx}}\left(\frac{t-\tau}{(t^{2\omega-1}-\tau^{2\omega-1}}\right)^{\frac{1}{2}}\right|$$

$$=\left\|\lim_{\substack{\tau\to t\\ \tau=tx}}\left(\frac{t-\tau}{(t^{2\omega-1}-\tau^{2\omega-1}}\right)^{\frac{1}{2}}=\lim_{x\to 1}\frac{1}{t^{\omega-1}}\left(\frac{1-x}{1-x^{2\omega-1}}\right)^{\frac{1}{2}}=\frac{(2\omega-1)^{-\frac{1}{2}}}{t^{\omega-1}}\right\|$$

$$=N_{1\omega}(t,\tau)\cdot\left|1-\frac{t^{\frac{\omega-1}{2}}}{\tau^{\frac{1-\omega}{2}}}\cdot\frac{1}{t^{\omega-1}}\right|=N_{1\omega}\left(t,\tau\right)\cdot\left|1-\frac{\tau^{\frac{\omega-1}{2}}}{t^{\frac{\omega-1}{2}}}\right|=N_{1\omega}(t,\tau)\cdot\left|\frac{t^{\frac{\omega-1}{2}}-\tau^{\frac{\omega-1}{2}}}{t^{\frac{\omega-1}{2}}}\right|$$

$$\leq(1-2\beta)\cdot\frac{t^{\frac{\omega-1}{2}}-\tau^{\frac{\omega-1}{2}}}{t-\tau}\frac{1}{t^{\frac{\omega-1}{2}}}\cdot\frac{t^{\omega}}{\tau^{\omega}}\cdot D\left[\frac{t^{\omega}\tau^{\omega}}{2a^2\left(t-\tau\right)}\right]^{\beta}\leq\left\|\frac{t^{\frac{\omega-1}{2}}-\tau^{\frac{\omega-1}{2}}}{t-\tau}<\frac{\omega-1}{2}t^{\frac{\omega-1}{2}-1}\right\|$$

$$\leq\frac{(1-2\beta)|\omega-1|t^{\frac{\omega-1}{2}-1}}{(2a^2)^{\beta}\cdot t^{\frac{\omega-1}{2}}}\cdot\frac{t^{\omega}}{\tau^{\omega}}\cdot\frac{t^{\beta\omega}\cdot\tau^{\beta\omega}}{(t-\tau)^{\beta}}=\frac{(1-2\beta)}{(2a^2)^{\beta}}\cdot|\omega-1|\cdot\frac{t^{\omega-1+\beta\omega}\cdot\tau^{-\omega+\beta\omega}}{(t-\tau)^{\beta}}$$

$$\leq C_1^{(1)}(\omega,\beta)\cdot(1-2\beta)\cdot|\omega-1|\cdot\frac{t^{\beta+1)\omega-1}\cdot\tau^{(\beta-1)\omega}}{(t-\tau)^{\beta}}.$$

Now, we estimate the second term. At first, we estimate

$$\left|1-e^{-(Q_h-Q_\omega)}\right|=\|Q_h\geq Q_\omega\|\leq Q_h(t,\tau)-Q_\omega(t,\tau)$$

$$=\frac{(2\omega-1)}{4a^2}(t^{2\omega-1}-\tau^{2\omega-1})-\frac{(t^{\omega}-\tau^{\omega})^2}{4a^2(t-\tau)}\leq\frac{(2\omega-1)}{4a^2}(t^{2\omega-1}-\tau^{2\omega-1}).$$

Then

$$N_{1h}(t,\tau)\cdot\left|1-e^{-(Q_h-Q_\omega)}\right|$$

$$\leq\frac{(1-2\beta)(2\omega-1)^2}{4a^2}\cdot\frac{t^{2\omega-1}(t^{2\omega-1}-\tau^{2\omega-1})}{\tau\cdot(t^{2\omega-1}-\tau^{2\omega-1})}\cdot e^{-Z}\cdot I_{\beta}(z)$$

$$\leq\left\|e^{-Z}\cdot I_{\beta}(z)<D_0z^{\beta}\right\|$$

$$\leq D_0\frac{(1-2\beta)(2\omega-1)^2}{4a^2}\cdot\frac{t^{2\omega-1}}{\tau}\cdot\frac{(2\omega-1)^{\beta}}{(2a^2)^{\beta}}\left[\frac{t^{2\omega-1}\cdot\tau^{2\omega-1}}{(t^{2\omega-1}-\tau^{2\omega-1})}\right]^{\beta}$$

$$\leq D_0\frac{(1-2\beta)(2\omega-1)^{2+\beta}}{4a^2\cdot(2a^2)^{\beta}}\cdot\frac{t^{2\omega-1}}{\tau}\cdot\left[\frac{t^{2\omega-1}\cdot\tau^{2\omega-1}}{(2\omega-1)\tau^{2\omega-2}(t-\tau)}\right]^{\beta}$$

$$=C_1^{(2)}(\beta,\omega)\cdot(1-2\beta)\cdot\frac{t^{(2\omega-1)}t^{(2\omega-1)\beta}\cdot\tau^{\beta-1}}{(t-\tau)^{\beta}}$$

$$=\left\|\begin{array}{c}(2\omega-1)\beta=(\beta\omega+\omega-1)-(1-\beta)(\omega-1);\\ t^{(2\omega-1)\beta}\cdot t^{-(1-\beta)(\omega-1)}\cdot\tau^{\beta-1}\leq t^{\omega-1+\beta\omega}\cdot\tau^{-\omega+\beta\omega}\end{array}\right\|$$

$$\leq C_1^{(2)}(\beta,\omega)\cdot t^{2\omega-1}\cdot\frac{t^{(\beta+1)\omega-1}\cdot\tau^{(\beta-1)\omega}}{(t-\tau)^{\beta}}.$$ □

The lemma is proved. Similarly, the validity of the following lemma is proved.

Lemma 7.3. *If* $\omega > \frac{1}{2}$, $0 < \beta < \frac{1}{2}$, *then the following inequality in valid:*

$$|P_{2\omega}(t,\tau) - P_{2h}(t,\tau)| \le C_2(\omega,\beta) \cdot \frac{t^{(\beta+1)\omega} \cdot \tau^{(\beta-1)\omega}}{(t-\tau)^{\beta}} \cdot e^{-Q_\omega(t,\tau)}.$$

Remark 7.2. *In the case* $\frac{1}{2} < \omega < 1$, *in the same inequalities it is sufficient to change the roles of the functions* $N_{i\omega}$ *and* N_{ih} $(i = 1,\ 2)$, *respectively.*

The proof of Theorem 7.1 follows from the Lemmas 7.1-7.3.

8. Solution of the characteristic equation

In order to find a solution to the characteristic equation, we make the following replacements:

$$t_1 = \left[\frac{1}{2\omega-1} \cdot t\right]^{\frac{1}{2\omega-1}}, \qquad \tau_1 = \left[\frac{1}{2\omega-1} \cdot \tau\right]^{\frac{1}{2\omega-1}},$$

then, we get

$$\mu_1(t_1) - \int\limits_0^t \left\{ \frac{(1-2\beta)\cdot t_1}{(t_1-\tau_1)\cdot\tau_1} \exp\left[-\frac{t_1\tau_1}{2a^2(t_1-\tau_1)}\right] \cdot I_\beta\left(\frac{t_1\tau_1}{2a^2(t_1-\tau_1)}\right) + \frac{{t_1}^2}{2a^2(t_1-\tau_1)^2} \exp\left[-\frac{t_1\tau_1}{2a^2(t_1-\tau_1)}\right] I_{\beta-1,\beta}\left(\frac{t_1\tau_1}{2a^2(t_1-\tau_1)}\right) + \frac{t_1}{2a^2(t_1-\tau_1)} \exp\left[-\frac{t_1\tau_1}{2a^2(t_1-\tau_1)}\right] I_\beta\left(\frac{t_1\tau_1}{2a^2(t_1-\tau_1)}\right) \right\} \mu_2(\tau_1)d\tau_1 = F(t_1),$$

the solution of which we obtained earlier [28]:

$$\mu_1(t_1) = F(t_1) + \int\limits_0^{t_1} R(t_1,\tau_1)F(\tau_1)d\tau_1 + C\mu_1^{(0)}(t_1).$$

where

$$\mu_1^{(0)}(t_1) = C \cdot \exp\left(-\frac{p_0(\beta)}{t_1}\right), \quad p_0(\beta) > 0.$$

and $p_0(\beta)$ – is the root of the equation

$$(1-2\beta)K_\beta\left(\frac{\sqrt{p}}{a}\right) - \frac{\sqrt{p}}{a}K_{\beta-1}\left(\frac{\sqrt{p}}{a}\right) = 0.$$

In this case, the resolvent $R(t_1,\tau_1)$ has the following estimate:

$$R(t_1,\tau_1) \le C(\beta)\frac{{t_1}^{1-(\beta-\varepsilon)} \cdot \tau_1^{-1-(\beta-\varepsilon)}}{(t_1-\tau_1)^{1-(\beta-\varepsilon)}}$$

Thus, the solution to the characteristic equation (9) has the form:

$$\mu_{1h}(t) = F_h(t) + \int\limits_0^t R_h(t,\tau)F_h(\tau)d\tau + C\mu_{1h}^{(0)}(t), \tag{10}$$

where the resolvent $R(t,\tau)$ satisfies the estimate:

$$R_h(t,\tau) \le C_1(\omega,\beta)\frac{t^{(2\omega-1)(1-(\beta-\varepsilon))} \cdot \tau_1^{-(2\omega-1)-(\beta-\varepsilon)}}{(t-\tau_1)^{1-(\beta-\varepsilon)}}, \quad 0 < \varepsilon < \beta \tag{11}$$

$$\mu_{1h}^{(0)}(t) = C \cdot \exp\left(-\frac{p_0(\beta)}{(2\omega-1)t^{2\omega-1}}\right), \quad p_0(\beta) > 0.$$

9. Regularization of the pseudo-Volterra integral equation (8) by solving the characteristic equation

Our goal is to solve the integral equation (8). We write the equation as:

$$\mu_1(t) - \int_0^t \left\{ \sum_{i=1}^{3} P_{i\omega}(t,\tau) \right\} \mu_1(\tau) d\tau = f_1(t), \tag{12}$$

$$P_{3\omega}(t,\tau) = N_{3\omega}(t,\tau) \cdot e^{-Q_\omega(t,\tau)}.$$

We regularize this equation by solving the characteristic equation

$$\mu_1(t) - \int_0^t \left\{ \sum_{i=1}^{2} P_{ih}(t,\tau) \right\} \mu_1(\tau) d\tau = f(t) + \int_0^t \left\{ \sum_{i=1}^{2} [P_{i\omega}(t,\tau) - P_{ih}(t,\tau)] + P_{3\omega}(t,\tau) \right\} \mu_1(\tau) d\tau.$$

Assuming the right-hand side is temporarily known, we write its solution and obtain the following integral equation:

$$\mathbb{K}_r \mu_1 \equiv \mu_1(t) - \int_0^t K(t,\tau) \mu_1(\tau) d\tau = f_{1r}(t), \tag{13}$$

where

$$K(t,\tau) = \sum_{i=1}^{2} [P_{i\omega}(t,\tau) - P_{ih}(t,\tau)] + P_{3\omega}(t,\tau)$$

$$+ \int_\tau^t R_h(t,\tau_1) \left\{ \sum_{i=1}^{2} [P_{i\omega}(\tau_1,\tau) - P_{ih}(\tau_1,\tau)] + P_{3\omega}(\tau_1,\tau) \right\} d\tau_1,$$

$$f_{1r}(t) = \int_0^t R_h(t,\tau_1) f_1(\tau_1) d\tau_1 + f_1(t) + C\mu_{1h}^{(0)}(t),$$

the resolvent $R_h(t,\tau_1)$ has the following estimate:

$$R_h(t,\tau_1) \le C_1(\omega,\beta,\gamma) \frac{t^{(2\omega-1)(1-\beta+\varepsilon)} \cdot \tau_1^{-(2\omega-1)-(\beta-\varepsilon)}}{(t-\tau_1)^{1-\beta+\varepsilon}}$$

where $0 < \varepsilon < \beta, \quad 0 < \beta < \frac{1}{2}$.

It is necessary to show that the integral equation (12) can be solved by the method of successive approximations. For this we need to estimate its kernel $K(t,\tau)$.

10. Estimates for the kernel of a regularized equation

Lemma 10.1. *The kernel of the regularized equation has the following estimate:*

$$|K(t,\tau)| \le C_1(\omega,\beta,\gamma)\cdot\left(\frac{t}{\tau}\right)^{(2\omega-1)(1-\beta+\varepsilon)}\cdot\frac{B(2\varepsilon\omega,1-\varepsilon)}{(t-\tau)^{\varepsilon}\cdot\tau^{1-2\varepsilon\omega}}.$$

where $B(a,b)$ – Beta function, $0<\varepsilon<\beta$, $0<\beta<\frac{1}{2}$.

Proof. Indeed, using the results of lemmas (7.1-7.3), we have:

$$\int_\tau^t R_h(t,\tau_1)\cdot\sum_{i=1}^{2}|N_{i\omega}(\tau_1,\tau)-N_{ih}(\tau_1,\tau)|d\tau_1$$
$$\le C_3(\omega,\beta,\gamma)\cdot t^{(2\omega-1)(1-\beta+\varepsilon)}\cdot\tau^{-\omega(1-\beta)}\cdot\int_\tau^t\frac{\tau_1^{-(2\omega-1)-(\beta-\varepsilon)}\cdot\tau_1^{\beta\omega-1+\omega}}{(t-\tau_1)^{1-\beta+\varepsilon}\cdot(\tau_1-\tau)^{\beta}}d\tau_1$$
$$\le C_3(\omega,\beta,\gamma)\cdot B(1-\beta;\beta-\varepsilon)\cdot\left(\tfrac{t}{\tau}\right)^{(2\omega-1)(1-\beta+\varepsilon)}\cdot\frac{1}{(t-\tau)^{\varepsilon}\cdot\tau^{1-2\omega\varepsilon}}.$$

Taking into account

$$N_{3\omega}(\tau_1,\tau)\le\frac{D_0\omega}{2a^2}\cdot\frac{\tau_1^{2\omega-1}}{\tau_1-\tau}\cdot\frac{a\sqrt{2}(\tau_1-\tau)^{\frac{1}{2}}}{\tau_1^{\frac{\omega}{2}}\cdot\tau^{\frac{\omega}{2}}}=\frac{D_0\omega}{a\sqrt{2}}\cdot\frac{\tau_1^{\frac{3}{2}\omega-1}}{(\tau_1-\tau)^{\frac{1}{2}}\cdot\tau^{\frac{\omega}{2}}},$$

we get:

$$\int_\tau^t R_h(t,\tau_1)\cdot N_{3\omega}(\tau_1,\tau)d\tau_1\le C_4\cdot B(\frac{1}{2},\ \beta-\varepsilon)\left(\frac{t}{\tau}\right)^{(2\omega-1)(1-\beta+\varepsilon)}\cdot\frac{\tau^{(2\omega-1)(\frac{1}{2}-\beta+\varepsilon)}}{(t-\tau)^{\frac{1}{2}-\beta+\varepsilon}\cdot\tau^{\frac{1}{2}+\beta-\varepsilon}}\ ,$$

Thus,

$$|K(t,\tau)| \le C_1(\omega,\beta,\gamma)\cdot\left(\frac{t}{\tau}\right)^{(2\omega-1)(1-\beta+\varepsilon)}\cdot\frac{B(2\varepsilon\omega,1-\varepsilon)}{(t-\tau)^{\varepsilon}\cdot\tau^{1-2\varepsilon\omega}}.$$

□

Remark 10.1. *From relation (12), it follows that a homogeneous equation*

$$\mu_1(t)-\int_0^t\left\{\sum_{i=1}^{3}N_{i\omega}(t,\tau)\right\}\ \mu_1(\tau)d\tau=0$$

is equivalent to an inhomogeneous equation

$$\mathbb{K}_r\mu_1=\mu_1(t)-\int_0^t K(t,\tau)\mu_1(\tau)d\tau=C\mu_{1h}^{(0)}(t),$$

which for each C has a unique solution, we denote it – $\widetilde{\mu}_1^{(0)}(t)$.

Thus the validity of the following theorem is proved.

Theorem 10.1. *For $0<\beta<\frac{1}{2}$, the general solution of the inhomogeneous integral equation (12) is the following function*

$$\mu_1(t)=[\mathbb{K}_r]^{-1}f(t)+C\cdot\widetilde{\mu}_1^{(0)}(t),\quad C=\text{const},$$

where

$$\mu_1(t)\in M(0,+\infty),\quad M(0,+\infty)=L_\infty(0,+\infty)\cap C(0,+\infty).$$

And for $\frac{1}{2} < \beta < 1$, it will have a unique solution, i.e. $C = 0$.

Therefore, taking into account (13), a solution to the original integral equation (6) has been found in the class of functions

$$t^{-\alpha+\omega(1-\beta)} e^{\frac{(2\omega-1)t^{2\omega-1}}{4a^2}} \mu t(t) \in L_\infty(0,\infty).$$

11. Solution of the boundary value problem (1-3)

From the integral representation for the solution (5) of the boundary value problem (1-3), we get

$$u(r,t) = \sum_{i=1}^{4} u_i(r,t),$$

where

$$u_1(r,t) = \int_0^t \frac{r^\beta \tau^{\omega(1-\beta)}(r-\tau^\omega)}{4a^4(t-\tau)^2} e^{-\frac{(r-\tau^\omega)^2}{4a^2(t-\tau)}} e^{-\frac{r\tau^\omega}{2a^2(t-\tau)}} I_\beta\left(\frac{r\tau^\omega}{2a^2(t-\tau)}\right)\mu(\tau)d\tau,$$

$$u_2(r,t) = \int_0^t \frac{r^{\beta+1} \tau^{\omega(1-\beta)}}{4a^4(t-\tau)^2} e^{-\frac{(r-\tau^\omega)^2}{4a^2(t-\tau)}} e^{-\frac{r\tau^\omega}{2a^2(t-\tau)}} I_{\beta-1,\beta}\left(\frac{r\tau^\omega}{2a^2(t-\tau)}\right)\mu(\tau)d\tau,$$

$$u_3(r,t) = \int_0^t \frac{r^\beta (1-2\beta)}{2a^2(t-\tau)\tau^{\omega\beta}} e^{-\frac{(r-\tau^\omega)^2}{4a^2(t-\tau)}} e^{-\frac{r\tau^\omega}{2a^2(t-\tau)}} I_\beta\left(\frac{r\tau^\omega}{2a^2(t-\tau)}\right)\mu(\tau)d\tau,$$

$$u_4(r,t) = \frac{1}{(2a^2)^\beta} \cdot \frac{1}{2^\beta} \cdot \frac{1}{\Gamma(\beta)} \int_0^t \frac{r^{2\beta}}{(t-\tau)^{\beta+1}} \cdot \exp\left[-\frac{r^2}{4a^2(t-\tau)}\right] \cdot g_1(t)d\tau.$$

For the solution $u(r,t) \in M(0,+\infty)$, we assume that $t^{-\alpha-\omega\beta} \cdot e^{\frac{(2\omega-1)t^{2\omega-1}}{4a^2}} \cdot \mu(t) \in L_\infty(0;+\infty)$, where $\alpha = (2\omega-1)(1-\beta+\varepsilon) > 0$. We estimate the first term:

$$u_1(r,t)$$
$$= \int_0^t \frac{r^\beta \tau^{\omega(1-\beta)}(r-\tau^\omega)\tau^{\alpha+\omega\beta}}{4a^4(t-\tau)^2} \; e^{-\frac{(r-\tau^\omega)^2}{4a^2(t-\tau)}} \; e^{-\frac{(2\omega-1)\tau^{2\omega-1}}{4a^2}} \; e^{-\frac{r\tau^\omega}{2a^2(t-\tau)}} I_\beta\left(\frac{r\tau^\omega}{2a^2(t-\tau)}\right)$$
$$\times \left\{\tau^{-\alpha-\omega\beta} \cdot e^{\frac{(2\omega-1)\tau^{2\omega-1}}{4a^2}} \cdot \mu(\tau)\right\} d\tau \le C_1 \int_0^t \frac{r^\beta \tau^{\omega+\alpha}(r-\tau^\omega)}{4a^4(t-\tau)^2} \; e^{-\frac{r\tau^\omega}{2a^2(t-\tau)}} I_\beta\left(\frac{r\tau^\omega}{2a^2(t-\tau)}\right) d\tau$$
$$= \left\| \frac{r\tau^\omega}{2a^2(t-\tau)} = z \Rightarrow dz = \frac{2a^2 r\left(\omega\tau^{\omega-1}(t-\tau)+\tau^\omega\right)}{4a^4(t-\tau)^2} \right\|$$
$$= C_1 \int_0^\infty \frac{r^\beta \tau^{\omega+\alpha}(r-\tau^\omega)}{4a^4(t-\tau)^2} \cdot \frac{4a^4(t-\tau)^2}{2a^2 r(\omega\tau^{\omega-1}(t-\tau)+\tau^\omega)} \cdot \frac{r\tau^\omega}{2a^2(t-\tau)} \cdot \frac{1}{z} e^{-z} I_\beta(z)\, dz$$
$$= \left\| r < t^\omega, \quad \tau < t, \quad \frac{\tau^\omega}{\omega\tau^{\omega-1}(t-\tau)+\tau^\omega} \le 1 \right\|$$
$$\le \frac{C_1}{4a^4} t^{\omega\beta} \int_0^\infty \frac{\tau^{\omega+\alpha}(t^\omega-\tau^\omega)}{t-\tau} \cdot \frac{1}{z} e^{-z} I_\beta(z) dz = \mathbb{I}^{(1)}(r,t).$$

Let $\omega > 1$, for which the inequality $t^\omega - \tau^\omega < \omega t^{\omega-1}(t-\tau)$ holds.

Then

$$\mathbb{I}^{(1)}(r,t) \leq \frac{C_1}{4a^4} t^{\omega\beta} \int_0^\infty \frac{\tau^{\omega+\alpha}\omega t^{\omega-1}(t-\tau)}{t-\tau} \cdot \frac{1}{z} e^{-z} I_\beta(z)dz = \| \ \tau < t \ \|$$

$$\leq \frac{C_1\omega}{4a^4} t^{\omega\beta+\alpha+2\omega-1} \int_0^\infty \frac{1}{z} e^{-z} I_\beta(z)dz = \frac{C_1\omega}{4a^4} t^{\omega\beta+\alpha+2\omega-1} \leq \widetilde{C_1}(a,T,\omega).$$

Let $\frac{1}{2} < \omega < 1$, for which the inequality $t^\omega - \tau^\omega < \omega\tau^{\omega-1}(t-\tau)$ holds.
Hence

$$\mathbb{I}^{(1)}(r,t)$$
$$\leq \tfrac{C_1}{4a^4} t^{\omega\beta} \int_0^\infty \tfrac{\tau^{\omega+\alpha}\omega\tau^{\omega-1}(t-\tau)}{t-\tau} \cdot \tfrac{1}{z} e^{-z} I_\beta(z)dz$$
$$= \tfrac{C_1\omega}{4a^4} t^{\omega\beta} \int_0^\infty \tau^{\alpha+2\omega-1} \cdot \tfrac{1}{z} e^{-z} I_\beta(z)dz = \left\| \begin{matrix} 2\omega-1>0 \quad \text{at } \frac{1}{2}<\omega<1 \\ \tau<t \end{matrix} \right\|$$
$$\leq \tfrac{C_1\omega}{4a^4} t^{\omega\beta+\alpha+2\omega-1} \int_0^\infty \tfrac{1}{z} e^{-z} I_\beta(z)dz = \tfrac{C_1\omega}{4a^4} t^{\omega\beta+\alpha+2\omega-1} \leq \widetilde{C_1}(a,T,\omega),\ \forall (r,t) \in G.$$

We estimate the second term:

$$u_2(r,t) = \int_0^t \frac{r^{\beta+1}\tau^{\omega(1-\beta)}\tau^{\alpha+\omega\beta}}{4a^4(t-\tau)^2} e^{-\frac{(r-\tau^\omega)^2}{4a^2(t-\tau)}} e^{-\frac{(2\omega-1)\tau^{2\omega-1}}{4a^2}} e^{-\frac{r\tau^\omega}{2a^2(t-\tau)}} I_{\beta-1,\beta}\left(\frac{r\tau^\omega}{2a^2(t-\tau)}\right)$$

$$\times \left\{ \tau^{-\alpha-\omega\beta} \cdot e^{\frac{(2\omega-1)\tau^{2\omega-1}}{4a^2}} \cdot \mu(\tau) \right\} d\tau \leq C_2 \int_0^t \frac{r^{\beta+1}\tau^{\omega+\alpha}}{4a^4(t-\tau)^2} e^{-\frac{r\tau^\omega}{2a^2(t-\tau)}} I_{\beta-1,\beta}\left(\frac{r\tau^\omega}{2a^2(t-\tau)}\right) d\tau$$

$$= \left\| \tfrac{r\tau^\omega}{2a^2(t-\tau)} = z \Rightarrow dz = \tfrac{2a^2r\left(\omega\tau^{\omega-1}(t-\tau)+\tau^\omega\right)}{4a^4(t-\tau)^2} \right\|$$
$$= C_2 \int_0^\infty \tfrac{r^{\beta+1}\tau^\alpha}{4a^4(t-\tau)^2} \cdot \tfrac{4a^4(t-\tau)^2}{2a^2r} \cdot \tfrac{\tau^\omega}{\omega\tau^{\omega-1}(t-\tau)+\tau^\omega} \cdot e^{-z} I_{\beta-1,\beta}(z)dz =$$
$$= \left\| r < t^\omega,\ \tfrac{\tau^\omega}{\omega\tau^{\omega-1}(t-\tau)+\tau^\omega} \leq 1 \ \right\|$$
$$\leq \tfrac{C_2}{2a^2} t^{\omega\beta+\alpha} \int_0^\infty e^{-z} I_{\beta-1,\beta}(z)dz = \tfrac{C_2}{2a^2} t^{\omega\beta+\alpha} \leq \widetilde{C_2}(a,T), \quad \forall (r,t) \in G.$$

Now, we will estimate the third term:

$$u_3(r,t) = \int_0^t \frac{r^\beta(1-2\beta)\tau^{\alpha+\omega\beta}}{2a^2(t-\tau)\tau^{\omega\beta}} e^{-\frac{(r-\tau^\omega)^2}{4a^2(t-\tau)}} e^{-\frac{(2\omega-1)\tau^{2\omega-1}}{4a^2}} e^{-\frac{r\tau^\omega}{2a^2(t-\tau)}} I_\beta\left(\frac{r\tau^\omega}{2a^2(t-\tau)}\right)$$

$$\times \left\{ \tau^{-\alpha-\omega\beta} \cdot e^{\frac{(2\omega-1)\tau^{2\omega-1}}{4a^2}} \cdot \mu(\tau) \right\} d\tau \leq C_3 \int_0^t \frac{r^\beta(1-2\beta)\tau^\alpha}{2a^2(t-\tau)} e^{-\frac{r\tau^\omega}{2a^2(t-\tau)}} I_\beta\left(\frac{r\tau^\omega}{2a^2(t-\tau)}\right) d\tau$$

$$= \left\| \tfrac{r\tau^\omega}{2a^2(t-\tau)} = z \Rightarrow dz = \tfrac{2a^2r\left(\omega\tau^{\omega-1}(t-\tau)+\tau^\omega\right)}{4a^4(t-\tau)^2} \right\|$$

$$= C_3 \int\limits_0^\infty \frac{r^\beta(1-2\beta)\tau^\alpha}{2a^2(t-\tau)} \cdot \frac{4a^4(t-\tau)^2}{2a^2 r(\omega\tau^{\omega-1}(t-\tau)+\tau^\omega)} \cdot \frac{r\tau^\omega}{2a^2(t-\tau)} \cdot \frac{1}{z} \cdot e^{-z} I_\beta(z)dz$$

$$= \left\| r < t^\omega, \quad \tau < t, \quad \frac{\tau^\omega}{\omega\tau^{\omega-1}(t-\tau)+\tau^\omega} \leq 1 \right\|$$

$$\leq \frac{C_3\,(1-2\beta)}{2a^2} t^{\omega\beta+\alpha} \int\limits_0^\infty \frac{1}{z} \cdot e^{-z} I_\beta(z)dz = \frac{C_3(1-2\beta)}{2a^2} t^{\omega\beta+\alpha} \leq \widetilde{C_3}(a,T,\beta), \qquad \forall (r,t) \in G.$$

And, finally, we will estimate the fourth term.

$$u_4(r,t) = \frac{1}{(2a^2)^\beta} \cdot \frac{1}{2^\beta} \cdot \frac{1}{\Gamma(\beta)} \int\limits_0^t \frac{r^{2\beta}}{(t-\tau)^{\beta+1}} \cdot \exp\left[-\frac{r^2}{4a^2(t-\tau)}\right] g_1(t)d\tau$$

$$\leq \frac{1}{(2a^2)^\beta} \cdot \frac{1}{2^\beta} \cdot \frac{1}{\Gamma(\beta)} \cdot |g_1(t)| \int\limits_0^t \frac{r^{2\beta}}{(t-\tau)^{\beta+1}} \cdot \exp\left[-\frac{r^2}{4a^2(t-\tau)}\right] d\tau$$

$$= \left\| \frac{r^2}{4a^2(t-\tau)} = z; \quad t-\tau = \frac{r^2}{4a^2 z}; \quad d\tau = \frac{r^2}{4a^2 z^2} dz \right\|$$

$$= \frac{1}{(2a^2)^\beta} \cdot \frac{1}{2^\beta} \cdot \frac{1}{\Gamma(\beta)} \cdot |g_1(t)| \int\limits_{\frac{r^2}{4a^2 t}}^\infty \frac{r^{2\beta}.4^{\beta+1}.a^{2\beta+2}.z^{\beta+1}}{r^{2\beta+2}} \cdot \frac{r^2}{4a^2 z^2} \cdot e^{-z} dz$$

$$= \frac{1}{(2a^2)^\beta} \cdot \frac{1}{2^\beta} \cdot \frac{1}{\Gamma(\beta)} \cdot 4^\beta \cdot a^{2\beta} \cdot |g_1(t)| \int\limits_{\frac{r^2}{4a^2 t}}^\infty z^{\beta-1} \cdot e^{-z} dz$$

$$= |g_1(t)| \cdot \frac{\Gamma(\beta, \frac{r^2}{4a^2 t})}{\Gamma(\beta)} < |g_1(t)|, \ \forall (r,t) \in G.$$

This implies the validity of Theorem (3.1) – the main result of the paper.

12. Conclusions

We studied a boundary-value problem for a heat-type equation in a time-dependent non-canonical domain whose spatial region collapses to a point at the initial time. By using generalized heat potentials, the problem was transformed into an explicit integral representation and reduced to a pseudo-Volterra integral equation of the second kind with a kernel of critical size, which makes it essentially different from the classical Volterra setting. For the parameter regime where the standard successive-approximation technique fails, we constructed an appropriate characteristic integral equation, obtained its explicit solution, and used it to regularize the original equation, derive kernel/resolvent estimates, and prove solvability in the class of bounded continuous functions, including a description of when uniqueness holds and when a nontrivial homogeneous component may appear. The developed approach is motivated by heat-transfer models with moving boundaries (e.g., in contact and arc devices) and provides a foundation for extensions to more general boundary motion laws.

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